The ADHM Construction and Anselmi's Topological Anomalies

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Abstract

We examine the anomalies arising in instanton calculus as detailed by Damiano Anslemi in 1994. Whereas Anselmi uses BRST theory, we use the ADHM construction to arrive at the same conclusions from a differential-geometric way. We observe that Anselmi's TQFT is similar to Donaldson Theory applied to charge 1 instantons on the 4-sphere,although the latter is really only used for 4-manifolds with $b_2^+ > 0$.

We demonstrate why the anomalies occur in the case of charge 1 instantons and move to show that similar anomalies cannot occur for instantons of higher charge. To do this, we develop an equivariant integration theory for hyperKahler manifolds and apply it to the hyperKahler geometry involved in the ADHM construction.

Contents

1 Introduction

In [1], Damiano Anselmi used BRST theory to discover certain topological anomalies arising in one of the more simple examples of TFT, namely the case of instantons, i.e bundles with ASD connections over S^4 (or $\mathbb{R}^4$ with certain certain restrictions on the choice of gauge). The moduli space of charge 1 instantons $\mathcal{M}_1$ (i.e bundles with $c_2 = 1$) of course is equivalent to 5 dimensional hyperbolic space which is contractible, hence all bundles over this space are trivial. As a result, all Donaldson polynomials formed from the cohomology of $\mathcal{M}_1$ actually vanish in this case.

Anselmi's method entailed building BRST representatives of the Donaldson μ cohomology classes of $\mathcal{M}_1$ from data on the moduli space of charge 1, and integrating these over certain submanifolds within $\mathcal{M}_1$. This process effectively calculates the Donaldson polynomial associated with these submanifolds. However, Anselmi's approach produced a non-trivial linking theory in certain dimensions when integrals were taken over certain "cycles"thus essentially contradicting the fact that there are no non-trivial Donaldson polynomials for S^4.

We use the ADHM construction as a basis for understanding the problem. This has the advantage that we can bypass BRST theory and obtain the same results in which Anselmi's Anomalies manifest themselves as singularities of Chern-Weil representatives of characteristic classes of the moduli space and show that in essence the anomalies arise due to the noncommutative procedure of removing unstable points and hyperKähler reduction.

We use an analogue of the Jeffrey-Kirwan localisation theorem for hyperKähler manifolds which relates the integration of equivariant cohomology classes over a hyperKähler manifold possessing a tri-Hamiltonian group action with the integration of cohomology classes on the quotient. We use this formula to show that there are no further anomalies for higher charge instantons essentially because of dimensional incompatibility.

1.1 Review of the ADHM construction

From the work of Penrose and the famous paper [3], it was shown that vector bundles with ASD connections could be constructed over S^4 using the conformal relationship betwen S^4 and $\mathbb{R}^4$ and little more than quaternionic linear algebra.

Recall that we may form ASD SU(2)-connections on $\mathbb{R}^4$ (S^4) by choosing

$$\begin{aligned}(T,P) &\in (i\mathfrak{u}(k)\otimes\mathbb{H})\oplus\left(\mathbb{C}^k\otimes_{\mathbb{R}}\mathbb{H}\right)\\ &\cong \left(i\mathfrak{u}(k)\oplus\mathbb{C}^k\right)\otimes_{\mathbb{R}}\mathbb{H}\\ &=: \mathfrak{M}^k_{\mathbb{C}},\end{aligned}$$

such that

$$\Im\left(T^*T+PP^*\right)=0$$

where $\Im$ denotes the quaternionic imaginary part which is well defined as $\mathfrak{M}^k_{\mathbb{C}}$ is very much a "quaternification" of a real vector space. We also require, for each $x\in\mathbb{R}^4$, that the map

$$\mathcal{R}_x=((T-x\mathbb{1})^*,P)\colon\mathbb{C}^k\otimes\mathbb{H}\oplus\mathbb{C}^k\longrightarrow\mathbb{C}^k\otimes\mathbb{H}$$

be surjective. Then we define a bundle $E=\ker\mathcal{R}$ and a connection given by $v^*\mathrm{d}v$, where

$$v_x=\begin{bmatrix}(x\mathbb{1}-T)^{*-1}P\\ \mathbb{1}\end{bmatrix}{\sigma_x}^{-1}$$

and

$${\sigma_x}^2=\mathbb{1}+P^*((T-x\mathbb{1})^*(T-x))^{-1}P.$$

The quaternionic column vector v forms the trivialisation of E over all of $\mathbb{R}^4$ (and hence S^4) except at the points where $x\in\mathbb{R}^4\cong\mathbb{H}$ is a left quaternionic eigenvalue for T. However such points are merely gauge singularities.

Since E is an SU(2)-bundle, we can simplify things a little here by converting the data into data on an Sp(1)-bundle. We can identify the fibre $\mathbb{C}^2$ with $\mathbb{H}$, so that as complex vector spaces

$$\begin{aligned}\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{H}) &\cong \mathrm{Hom}(\mathbb{H},\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{H})\\ &\cong \mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{H}\otimes_{\mathbb{C}}\mathbb{H}\\ &\cong \mathbb{C}^k\otimes_{\mathbb{C}}(\mathbb{H}\otimes_{\mathbb{R}}\mathbb{C})\\ &\cong \mathbb{C}^k\otimes_{\mathbb{R}}\mathbb{H}\\ &\cong \mathbb{H}^k\otimes_{\mathbb{R}}\mathbb{C}.\end{aligned}$$

This means that $\mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^k\otimes_{\mathbb{C}}\mathbb{H})$ may be regarded as the complexification of the real space

$$\left(\mathbb{R}^4\right)^k\cong\mathbb{H}^k.$$

So,using this conversion, we can recover ASD connections by choosing

$$(T,P) \in \left(\odot^2(\mathbb{R}^k) \otimes \mathbb{H}\right) \oplus \mathbb{H}^k =: \mathfrak{M}^k_{\mathbb{R}}$$

subject to the ADHM condition

$$\Im\left(T^*T + PP^*\right) = 0 \tag{1}$$

and the nondegeneracy condition,

$$\mathcal{R}_x \text{ is surjective for all } x \in \mathbb{R}^4. \tag{2}$$

We regard T as a symmetric matrix and P as a column vector each with quaternionic coefficients. Now from a result by Wood [11], every quaternionic matrix has a left quaternionic eigenvalue. This means that $(T,0)$ always gives a reducible solution, i.e $(T,0)$ does not satisfy (2).

We set $\mathbf{A}(k) \subset \mathfrak{M}^k_{\mathbb{R}}$ to be the elements that satisfy (1) and define $\mathbf{A}^*(k)$ to be the set of all elements of $\mathbf{A}(k)$ that satisfy the nondegeneracy condition (2).

2 Group actions and Moduli

It is a well known result that factoring out certain group actions from $\mathfrak{M}^k_{\mathbb{C}}$ or $\mathfrak{M}^k_{\mathbb{R}}$ we recover precisely the moduli space of instantons of charge k. Essentially this says that an orbit of ADHM data with respect to a certain group yields a gauge equivalence class of instantons.

For $\mathfrak{M}^k_{\mathbb{R}}$ this group is $\O k \times \mathrm{Sp}(1)$ which acts on $\mathfrak{M}^k_{\mathbb{R}}$ via

$$(\alpha,\beta):\ (T,P) \longrightarrow (\alpha T\alpha^{-1}, \alpha P\beta^{-1}).$$

It isn't hard to show that in fact

$$\Im(T^*T + PP^*)$$

is invariant with respect to the Sp(1) action and equviariant with respect to the $\O k$ action. In fact

$$\vec{\mu}(T,P) = \Im(T^*T + PP^*)$$

is the HyperKähler moment map on $\mathfrak{M}^k_{\mathbb{R}}$ with its obvious hyperKähler structure with respect to the action of $\O k$.

We fix the canonical basis of $\mathbb{H}$ by

$$\begin{aligned} q_0 &= \mathbb{1} \\ q_1 &= \widehat{\mathbf{i}} \\ q_2 &= \widehat{\mathbf{j}} \\ q_3 &= \widehat{\mathbf{k}}. \end{aligned}$$

Proposition 2.1 *The action of* $\mathrm{Sp}(1)$ *is free on* $\mathbf{A}^*(k)$*.*

Proof
Suppose there is $\beta \in \mathrm{Sp}(1)$ with $P\beta = P$. Since we require $(T,P) \in \mathbf{A}^*(k)$,we are assured that $P \neq 0$. Thus $P^\mu\beta = P^\mu$ for some nonzero component $P^\mu \in \mathbb{H}$ of P. Hence $\beta = 1$. ■

To show that $\emptyset k$ acts freely on $\mathbf{A}^*(k)$, we need the following result.

Lemma 2.2 *If* $(T,P) \in \mathfrak{M}_{\mathbb{R}}^k$ *is fixed by* $u \in \emptyset k$ *then there is a decomposition*

$$T = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix} \quad P = \begin{pmatrix} P' \\ 0 \end{pmatrix}$$

with $(T',P') \in \mathfrak{M}_{\mathbb{R}}^l$ *and* $T'' \in \odot^2(\mathbb{R}^{k-l}) \otimes \mathbb{H}$*. Hence such* $(T,P) \notin \mathbf{A}^*(k)$*.*

Proof
Suppose $u \in \emptyset k$ fixes (T,P).

Then $uP = P$, and $P = P_i q_i$ for vectors $P_i \in \mathbb{R}^k$, and $uP_i = P_i$ for each $i = 0...3$. Hence u has at least one eigenvector with eigenvalue 1. Decompose $\mathbb{R}^k$ into $V \oplus W$ where V is the maximal 1-eigenspace of u and W its orthogonal complement. Notice that $V \supset \mathrm{Span}\{P_i | i = 0...3\}$. With respect to this decomposition, we have

$$T = \begin{pmatrix} T' & T_0^\top \\ T_0 & T'' \end{pmatrix}, \quad P = \begin{pmatrix} P' \\ 0 \end{pmatrix},$$

and

$$u = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & u' \end{pmatrix},$$

for some $u' \in \emptyset k - l$. Assume that $u \neq \mathbb{1}$ and hence $W \neq 0$. The condition that $uTu^{-1} = T$ shows us, in particular that $u'T_0 = T_0$, so the columns of T_0, T_0^μ say, satisfy

$$u'T_0^\mu = T_0^\mu$$

But u' does not have $+1$ as an eigenvalue by the decomposition. Hence $T_0^{\mu} = 0$ and $T_0 = 0$.

Thus we have the following decompositions for T and P with respect to this splitting.

$$T = \begin{pmatrix} T' & 0 \\ 0 & T'' \end{pmatrix} \quad P = \begin{pmatrix} P' \\ 0 \end{pmatrix}.$$

Now by R. Wood [11], we know that T'' (hence T) has a left eigenvalue $\lambda \in \mathbb{H}$. So

$$\mathcal{F}_{(\lambda,T,P)} = \mathcal{R}_{(\lambda,T,P)}\mathcal{R}^*_{(\lambda,T,P)}$$

will not be invertible and thus $(T, P) \notin \mathbf{A}^*(k)$.

■

Corollary 2.3 *The following are equivalent*

1. (T, P) *satisfies the nondegeneracy condition;*
2. (T, P) *has trivial stabiliser under* $\text{Ø}k$.

Corollary 2.4 *The action of* $\text{Ø}k$ *is free on* $\mathbf{A}^*(k)$.

We may have a small problem here. Although $\text{Ø}k$ and $\mathrm{Sp}(1)$ individually act freely on $\mathbf{A}^*(k)$, the full group $\text{Ø}k \times \mathrm{Sp}(1)$ doesn't act freely here. Each point is fixed by $\pm(\mathbb{1}, 1)$ meaning that the group of symmetries we require is

$$\frac{\text{Ø}k \times \mathrm{Sp}(1)}{\mathbb{Z}_2}.$$

Not only that but for $k \geq 2$, we have some $(T, P) \in \mathfrak{M}^k_{\mathbb{R}}$ such that

$$gTg^{-1} = T, \quad gPq^{-1} = P$$

for certain $(g, q) \in \text{Ø}k \times \mathrm{Sp}(1)$. We must check that they are not in $\mathbf{A}^*(k)$.

2.1 The Action of $\text{Ø}k \times \mathrm{Sp}(1)$ on $\mathfrak{M}^k_{\mathbb{R}}$

Choose $\xi \in \text{ø}k$, and choose a basis of $\mathbb{R}^k$ (hence of $\mathbb{H}^k$) in which

$$\xi = \begin{pmatrix} 0 & \mathbb{1}_l & 0 \\ -\mathbb{1}_l & 0 & 0 \\ 0 & 0 & 0_{k-2l} \end{pmatrix}$$

where $\mathbb{1}_l$ is the $l \times l$ identity matrix and 0_{k-2l} is the $(k-2l) \times (k-2l)$ zero matrix. It may happen that $k = 2l$, in which case ξ is invertible and forms a complex structure on $\mathbb{R}^k = \mathbb{R}^{2l}$. Thus for

$$T = \begin{pmatrix} T' & T_1 & T_2^\top \\ T_1^\top & T'' & T_3^\top \\ T_2 & T_3 & T''' \end{pmatrix}, \quad P = \begin{pmatrix} P' \\ P'' \\ P''' \end{pmatrix}$$

we find

$$[\xi, T] = \begin{pmatrix} T_1 + T_1^\top & T'' - T' & T_3^\top \\ T' - T'' & -T_1 - T_1^\top & -T_2^\top \\ T_3 & -T_2 & 0 \end{pmatrix}, \quad \xi P = \begin{pmatrix} -P'' \\ P' \\ 0 \end{pmatrix}$$

So if $[\xi, T] = 0$ and $\xi P = P\alpha$ then

$$T = \begin{pmatrix} T' & T_1 & 0 \\ -T_1 & T' & 0 \\ 0 & 0 & T''' \end{pmatrix}, \quad P = \begin{pmatrix} P' \\ P'\alpha \\ 0 \end{pmatrix}$$

with the condition that $|\alpha| = 1$. From this we can conclude, that if (T, P) satisfies the ADHM condition then it is a reducible solution, unless ξ is invertible and hence k is even.

Let us now suppose that ξ is invertible, then as we mentioned, ξ is a complex structure on $\mathbb{R}^{2l}$. Let us take the complex point of view and look at the process for $\mathbb{C}^l$. Under this identification ξ becomes multiplication by i and

$$T = T' + iT'', P = P' + iP''.$$

Also our condition that $\xi P = P\alpha$ becomes

$$iP = P\alpha$$

or

$$-P' + iP' = P'\alpha + iP'\alpha.$$

Comparing the real and imaginary parts,

$$\begin{aligned} P'\alpha &= -P', \\ P'\alpha &= P', \end{aligned}$$

since α is considered to have only real coefficients. Hence $P' = 0$ and $P = 0$, and the solution is again reducible.

So far, we have proved that if the vector field induced by $(\xi, \alpha) \in øk \oplus \mathfrak{sp}(1)$ vanishes at a point (T, P) then (T, P) is a reducible solution. This in turn shows that the stabiliser of $(T, P) \in \mathbf{A}^* (k)$ must be a discrete, hence finite subgroup of $Øk \times \mathrm{Sp}(1)$ which is enough for our purposes.

2.2 Introducing the Moduli Spaces

Let

$$\tilde{M}_k = \frac{\mathbf{A}^*(k)}{\emptyset k}$$

and

$$\begin{aligned} M_k &= \tilde{M}_k / \left(\frac{\mathrm{Sp}(1)}{\mathbb{Z}_2} \right) \\ &= \tilde{M}_k / \mathrm{SO}(3). \end{aligned}$$

Define the Atiyah map $\mathrm{At}\colon \mathbf{A}^*(k) \longrightarrow \mathcal{A}_k$ where $\mathcal{A}_k$ is the space of connections of charge k by

$$(T, P) \mapsto v^* \mathrm{d}v$$

as above. It is well known that any two elements of the same orbit produce gauge equivalent connections, and that M_k is diffeomorphic to $\mathcal{M}_k$, the moduli space of ASD connections of charge k[9].

Also, by considering $\tilde{M}_k = \frac{\mathbf{A}^*(k)}{\emptyset K}$, we obtain the framed moduli space of connections. The manifold of equivalence classes of ADHM data under the action of $\emptyset k$ is precisely the moduli space of framed instantons, $\widetilde{\mathcal{M}}_k$.

2.3 Curvature of bundles under group actions

Here, we recall the theory of vector bundles and group actions detailed in section 5.2.3 of [6]. Let $\widehat{\pi}\colon \widehat{E} \longrightarrow \widehat{Y}$ be a vector bundle, and G a Lie group whose action on $\hat{E}$ covers a free action on $\widehat{Y}$. Also let $\widehat{E}$ be endowed with a G-invariant connection $\widehat{\nabla}$. Our aim is to construct a connection ∇ on the factor bundle $E = \frac{\widehat{E}}{G}$ over $Y = \frac{\widehat{Y}}{G}$.

To do this we need a connection on the principal G-bundle $p : \widehat{Y} \longrightarrow Y$. This will enable us to lift tangent vectors on Y to $\widehat{Y}$ and compute directional derivatives. We suppose that it is given in the form of a horizontal distribution $H \subset \mathrm{T}\widehat{Y}$, with connection 1-form θ.

Any section $s \in Y; E0$ comes from an invariant section $\widehat{s} \in \widehat{Y}; \widehat{E}0$. Therefore we can set

$$\widehat{\nabla_X s} = \widehat{\nabla}_{\widehat{X}} \widehat{s},$$

where $\widehat{X}$ is the horizontal lift of X. This descends to the quotient so that $\widehat{\nabla_X s}$ is the lift of the object which can be called $\nabla_X s$.

From the definition of Y, we know that $p^*E \cong \widehat{E}$, so we can consider the effect of the pull back $p^*\nabla$ on $\widehat{E}$. Now, the directional derivatives of $p^*\nabla$ will vanish on vertical vectors in $\mathrm{T}\widehat{Y}$, hence we know that

$$\widehat{\nabla} = p^*\nabla + V$$

where V vanishes on horizontal vectors, and is equivariant under the G action by definition of $\widehat{\nabla}$.

It is therefore obvious that $V = \Psi\theta$ where Ψ is a linear $\mathfrak{g} \longrightarrow \mathrm{End}(\widehat{E})$

It can then be shown (and [6] do this to some extent) that

$$F(\widehat{\nabla})(\widehat{X_1}, \widehat{X_2}) = F(\nabla)(X_1, X_2) + \Psi F(\theta)(\widehat{X_1}, \widehat{X_2}), \tag{3}$$

allowing us to compare the curvature of $\widehat{\nabla}$ with ∇.

3 The case of charge 1 instantons $(k = 1)$

For $k = 1$, we are in a truly interesting position since for any

$$(T, P) \in \mathbf{A}^*(1) \subset \mathfrak{M}^1_{\mathbb{R}} = \left(\odot^2(\mathbb{R}) \otimes \mathbb{H}\right) \oplus \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H},$$

the ADHM condition

$$\Im(T^*T + PP^*) = 0$$

is automatically satisfied, hence $\mathbf{A}(1) = \mathfrak{M}^1_{\mathbb{R}}$.

Now we can construct a canonical bundle $\widehat{E} \longrightarrow \mathbf{A}(1) \times \mathbb{R}^4$ as follows.
For each $(T, P, x) \in \mathbf{A}(1) \times \mathbb{R}^4$, define the fibre of $\widehat{E}$ to be

$$\widehat{E}_{(T,P,x)} = \ker \mathcal{R}_x = \ker(T - x, P) : \mathbb{H} \oplus \mathbb{H} \longrightarrow \mathbb{H}.$$

We can define a connection $\widehat{\nabla}$ on $\widehat{E}$ given at the point (T, P, x) by $\mathrm{At}(T, P)_x$. From the suggestive terminology, it is clear that we will choose $\widehat{Y} = \mathbf{A}^*(1) \times \mathbb{R}^4$. Since we have removed the singular points, the action of

$$\frac{\varnothing 1 \times \mathrm{Sp}(1)}{\mathbb{Z}_2} = \frac{\mathbb{Z}_2 \times \mathrm{Sp}(1)}{\mathbb{Z}_2} = \mathrm{Sp}(1)$$

is free. It is also clear that by construction $\widehat{\nabla}$ is $\mathrm{Sp}(1)$ invariant.

3.1 The curvature of the $\mathrm{Sp}(1)$-bundle

We now need to consider the principal Sp(1)-bundle

$$\mathbf{A}^*(1) \longrightarrow M_1.$$

For $(T,P) \in \mathbf{A}^*(1)$, the action of the group Sp(1) is

$$u : (T,P) \mapsto (T, Pu^{-1}).$$

The vertical subspace $V_{(T,P)}$ will therefore be

$$\{(0, -P\widehat{u}) | \widehat{u} \in \mathfrak{sp}(1)\}.$$

We define a horizontal subspace $H_{(T,P)}$ to be the orthogonal complement of the vertical subspace. Thus $H_{(T,P)}$ will be the space

$$\{(t,p) | P^*p \in \mathbb{R}\} = \ker(p \mapsto \Im(P^*p))\}.$$

This immediately gives us the connection 1-form

$$\theta_{(T,P)}(t,p) = -\frac{1}{|P|^2}\Im(P^*p). \tag{4}$$

Proposition 3.1 *θ is indeed a connection 1-form.*

Proof
First, θ is well defined as $P \neq 0$.

On vertical vectors

$$\begin{aligned}
\theta_{(T,P)}(0, -P\widehat{u}) &= -\frac{1}{|P|^2}\Im(-P^*P\widehat{u}) \\
&= \Im(\widehat{u}) \\
&= \widehat{u}.
\end{aligned}$$

Also, for equivariance,

$$\begin{aligned}
\theta_{(T,Pu^{-1})}(t, pu^{-1}) &= -\frac{1}{|Pu|^2}\Im(uP^*pu^{-1}) \\
&= -\frac{1}{|P|^2}u\Im(P^*p)u^{-1} \\
&= \mathrm{ad}(u)\theta_{(T,P)}(t,p).
\end{aligned}$$

$\blacksquare$

Using local coordinates (T, P), this connection 1-form can be written

$$\theta = -\frac{1}{|P|^2}\Im(P^*\mathrm{d}P),$$

so the curvature form restricted to the horizontal space is given by

$$\begin{aligned} F(\theta)\Big|_{\ker\theta} &= \mathrm{d}\theta\Big|_{\ker\theta} \\ &= -\frac{1}{|P|^2}\mathrm{d}P^* \wedge \mathrm{d}P\Big|_{\ker\theta} - \frac{1}{|P|^2}\mathrm{d}|P|^2 \wedge \theta\Big|_{\ker\theta} \\ &= -\frac{1}{|P|^2}\mathrm{d}P^* \wedge \mathrm{d}P\Big|_{\ker\theta}. \end{aligned}$$

Now $(t, p) \in \mathbf{A}\,(1)$ is horizontal if and only if $p = \lambda P$ by the definition of θ in (4). Hence for a horizontal vector $(X, t, \lambda P) \in \mathbb{R}^4 \times H_{(T,P)}$

$$\mathrm{d}P((X, t, \lambda P)) = \lambda P,$$

and from this we can see that

$$\mathrm{d}P\Big|_{\ker\theta} = \frac{\delta|P|^2}{2|P|^2}P = \frac{\delta|P|}{|P|}\Big|_{\ker\theta}P. \tag{5}$$

As a result

$$F(\theta)\Big|_{\ker\theta} = -\delta|P| \wedge \delta|P|\Big|_{\ker\theta} = 0.$$

This means that the formula (3) becomes

$$F(\nabla)(X_1, X_2) = F(\widehat{\nabla})(\widehat{X_1}, \widehat{X_2}).$$

3.2 The Curvature of the Universal Bundle

Recall that the Atiyah map was defined by

$$\begin{gathered} \mathrm{At}\colon \mathbf{A}\,(1) \longrightarrow \mathcal{A}_k \\ (T, P) \mapsto v(T, P)^*\mathrm{d}v(T, P). \end{gathered}$$

Explicitly for $k = 1$, $\mathrm{At}(T, P)$ is given by $v(T, P)^*\mathrm{d}v(T, P)$ where

$$v(T, P) = \frac{|x - T|}{\sqrt{|P|^2 + |x - T|^2}}\begin{bmatrix} \frac{(x-T)}{|x-T|^2}P \\ 1 \end{bmatrix}.$$

Thus $\widehat{\nabla}$ is given at the point (T, P, x) by $v(T,P)_x^* \widehat{\mathrm{d}} v(T,P)_x$, where $\widehat{\mathrm{d}}$ denotes the de Rham differential not only with respect to x but also with respect to the quaternions T and P. We will also denote by δ the de Rham differential in the ADHM space. Also, set $\tilde{x} = x - T$

Expanding this calculation and setting

$$\Delta = \sqrt{|P|^2 + |\tilde{x}|^2},$$

we find that the connection 1-form of $\widehat{\nabla}$ with respect to the global trivialisation of $\widehat{E}$ is

$$v(T,P)_x^* \widehat{\mathrm{d}} v(T,P)_x = \frac{1}{\Delta^2} \Im \left(P^* \mathrm{d}P - \frac{1}{|\tilde{x}|^2} P^* \widehat{\mathrm{d}} \tilde{x}^* \tilde{x} P \right). \tag{6}$$

We compute the curvature in the obvious way, i.e.

$$F(A) = \mathrm{d}A + A \wedge A.$$

From this we get

$$\begin{aligned} F(\widehat{\nabla}) = \tfrac{1}{\Delta^4} \Big[& |\tilde{x}|^2 \mathrm{d}P^* \wedge \mathrm{d}P + \tfrac{1}{|\tilde{x}|^2} P^* \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} \wedge \widehat{\mathrm{d}} \tilde{x}^* \tilde{x} P \\ & - P^* \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} \wedge \mathrm{d}P - \mathrm{d}P^* \wedge \widehat{\mathrm{d}} \tilde{x}^* \tilde{x} P \Big]. \end{aligned} \tag{7}$$

Now, we are interested in "factoring" out $\widehat{\nabla}$ by Sp(1). That is, we want only to consider the horizontal lifts of tangent vectors and vector fields on M_1 and how $F(\widehat{\nabla})$ behaves when restricted to these.

So by (5), we have

$$\begin{aligned} & F(\widehat{\nabla}) \Big|_{\ker \theta} \\ = \; & \frac{1}{\Delta^4} \left[|\tilde{x}|^2 \delta |P| \wedge \delta |P| + \frac{1}{|\tilde{x}^2|} P^* \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} \wedge \widehat{\mathrm{d}} \tilde{x}^* \tilde{x} P - 2 \Im (P \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} \wedge \frac{\delta |P|}{|P|} P) \right] \\ = \; & \frac{1}{\Delta^4} \left[\frac{1}{|\tilde{x}^2|} P^* \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} \wedge \widehat{\mathrm{d}} \tilde{x}^* \tilde{x} P + 2 \frac{\delta |P|}{|P|} \wedge \Im (P \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} P) \right]. \end{aligned}$$

If we set $|P| = \rho$, then we find that when restricted to the horizontal space

$$F(\widehat{\nabla}) = \frac{1}{\Delta^4} \left[\frac{1}{|\tilde{x}^2|} P^* \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} \wedge \widehat{\mathrm{d}} \tilde{x}^* \tilde{x} P + 2 \frac{\delta \rho}{\rho} \wedge \Im (P \tilde{x}^* \widehat{\mathrm{d}} \tilde{x} P) \right]. \tag{8}$$

This is very similar to Anselmi's formula in [1]. In fact if we assume, (as he effectively does) that P is real and positive under part of a gauge fixing condition, then we have his answer scaled by a factor of $\frac{1}{2}$.

3.3 Calculating the μ map

Our next goal is to compute a form representing the second Chern class of E. This will be the same as the second Chern character $\mathrm{ch}_2(E)$ since the group is $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ and hence $\mathrm{c}_1(E) = 0$. We may obtain a representative of the cohomology class of $c_2(E)$ given by

$$\frac{1}{4\pi^2}\mathrm{tr}\left(F(\nabla) \wedge F(\nabla)\right).$$

By expanding this we find that the desired form is

$$c = \frac{6\rho^4}{\Delta^8\pi^2}\widehat{\mathrm{d}}\tilde{x}_1 \wedge \widehat{\mathrm{d}}\tilde{x}_2 \wedge \widehat{\mathrm{d}}\tilde{x}_3 \wedge \widehat{\mathrm{d}}\tilde{x}_4 - \frac{6\rho^3}{\Delta^8\pi^2}\delta\rho \wedge \sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4 \quad (9)$$

where $\tilde{x}_i = x_i - T_i$.

Now, it can be checked that

$$c = \widehat{\mathrm{d}}\left(\frac{1}{2\pi^2}\frac{(|\tilde{x}|^2 + 3\rho^2)}{(|\tilde{x}|^2+\rho^2)^3}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4\right).$$

We choose a compact d-dimensional submanifold Σ of $\mathbb{R}^4$ and let $\alpha \in \Omega^{4-d}_{cpt}(\mathbb{R}^4)$ be a closed form such that for any $\beta \in \Sigma d$.

$$\int_\Sigma \beta = \int_{\mathbb{R}^4} \beta \wedge \alpha.$$

We also choose a submanifold Ξ of M_1 whose intersection with ∂M_1 is compact.

$$\begin{aligned}
\int_{\Xi\times\Sigma} c &= \int_{\Xi\times\mathbb{R}^4} c \wedge \alpha \\
&= \int_{(\partial M\cap\Xi)\times\mathbb{R}^4}\left(\frac{1}{2\pi^2}\frac{(|\tilde{x}|^2 + 3\rho^2)}{(|\tilde{x}|^2+\rho^2)^3}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4\right) \wedge \alpha \\
&= \frac{1}{2\pi^2}\int_{(\Xi\cap\{\rho=0\})\times\mathbb{R}^4}\frac{1}{|\tilde{x}|^4}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4 \wedge \alpha \\
&= \frac{1}{2\pi^2}\int_{(\Xi\cap\{\rho=0\})\times\Sigma}\frac{1}{|\tilde{x}|^4}\sum_{i=1}^{4}(-1)^i\tilde{x}_i\widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4.
\end{aligned}$$

The right hand side above can be seen to be the Gauß formula for the linking number of Σ with $\Xi \cap \{P = 0\}$ regarded as a 3-dimensional submanifold of $\mathbb{R}^4$.

3.4 The Donaldson Polynomials in the case $k = 1$

We now have to assess the consequences of this for the Donaldson Polynomials. Let $\Sigma_1, \ldots, \Sigma_d$ be d submanifolds of $\mathbb{R}^4$. We may form the Donaldson μ class

$$\mu_{Don}(\Sigma_i) = \int_{\Sigma_i} c$$

and from this the Donaldson polynomial

$$\begin{aligned} \mathrm{Don}_1(\Sigma_1, \ldots, \Sigma_d) &= \int_{\mathcal{M}_1} \mu_{Don}(\Sigma_1) \ldots \mu_{Don}(\Sigma_d) \\ &= \int_{\mathcal{M}_1 \times \Sigma_1 \times \ldots \times \Sigma_d} c_1 \wedge \ldots \wedge c_d \end{aligned}$$

where c_i is c restricted to Σ_i.

Now let

$$f(x, T, \rho) = \frac{1}{2\pi^2} \frac{(|\tilde{x}|^2 + 3\rho^2)}{(|\tilde{x}|^2 + \rho^2)^3}$$

so that

$$c = \widehat{\mathrm{d}}\,(f(x, T, P)\alpha(x, T, P))$$

where

$$\alpha(x, T, P) = \sum_{i=1}^{4} (-1)^i \tilde{x}_i \widehat{\mathrm{d}}\tilde{x}_1 \ldots \wedge \hat{i} \wedge \ldots \wedge \widehat{\mathrm{d}}\tilde{x}_4.$$

Then

$$\begin{aligned} & \mathrm{Don}_1(\Sigma_1, \ldots, \Sigma_d) \\ = & \int_{\mathcal{M}_1 \times \Sigma_1 \times \ldots \times \Sigma_d} c_1 \wedge \ldots \wedge c_d \\ = & \int_{\mathcal{M}_1 \times \Sigma_1 \times \ldots \times \Sigma_d} \widehat{\mathrm{d}}(f(x^1, T, P)\alpha(x^1, T, P)) \wedge \ldots \wedge \widehat{\mathrm{d}}(f(x^d, T, P)\alpha(x^d, T, P)) \\ = & \int_{\mathcal{M}_1 \times \Sigma_1 \times \ldots \times \Sigma_d} \widehat{\mathrm{d}} \left(f(x^1, T, P)\alpha(x^1, T, P) \wedge \ldots \wedge \widehat{\mathrm{d}}(f(x^d, T, P)\alpha(x^d, T, P)) \right) \\ = & \int_{\{\rho=0\} \times \Sigma_1 \times \ldots \times \Sigma_d} f(x^1, T, P)\alpha(x^1, T, P) \wedge \ldots \wedge \widehat{\mathrm{d}}(f(x^d, T, P)\alpha(x^d, T, P)) \\ = & \frac{1}{2\pi^2} \lim_{\rho \longrightarrow 0} \int_{\{(T,\rho)\} \times \Sigma_1 \times \ldots \times \Sigma_d} \frac{(3\rho^2 + |x^1 - T|^2)}{(\rho^2 + |x^1 - T|^2)^3} \alpha(x^1, T, 0) \wedge \\ & \prod_{l=1}^{d} \left(\frac{6\rho^4}{(\rho^2 + |x^l - T|^2)^4} \widehat{\mathrm{d}}(x^l_1 - T_1) \wedge \ldots \widehat{\mathrm{d}}(x^l_4 - T_4) \right) \end{aligned}$$

using the formula (9) for c.

Now, since we have a singularity when $x = T$ and $\rho = 0$ in the formula

$$\frac{6\rho^4}{(\rho^2 + |x - T|^2)^4}$$

we have to be a bit careful with limits. Now, a calculation shows that

$$\lim_{\rho \longrightarrow 0} \frac{6\rho^4}{(\rho^2 + |x - T|^2)^4} = \frac{1}{2} \mathrm{vol}\, \mathrm{S}^3 \delta(x - T)$$

and provided $x^1, \ldots, x^d$ are distinct points in $\mathbb{R}^4$

$$\lim_{\rho \longrightarrow 0} \prod_{l=1}^{d} \frac{6\rho^4}{(\rho^2 + |x^l - T|^2)^4} = \prod_{l=1}^{d} \frac{1}{2} \mathrm{vol}\, \mathrm{S}^3 \delta(x^l - T).$$

For $d = 2$, we have

$$\mathrm{Don}_1(\Sigma_1, \Sigma_2) = \frac{\mathrm{vol}\, \mathrm{S}^3}{2} \int_{\Sigma_1} \int_{\Sigma_2} \frac{1}{|x^1 - x^2|^2} \alpha(x^1, x^2, 0)$$

which is a constant multiple of the linking number of Σ_1 with Σ_2.

For $d > 2$, the situation is much more complicated. A discussion of this can be found in another article by Anselmi, [2].

Although Donaldson theory is relevant only for 4-manifolds with $b^2_+ > 0$, the formation of μ-classes is completely trivial on S^4, and one would expect all polynomials constructed with Chern-Weil representatives of these μ-classes as described above to agree with this triviality. However, Anselmi has produced "representatives" of μ-classes which form non-trivial polynomials, contradicting our expectations.

3.5 Reasons for the Anomalies

Let us examine what is going on more closely. The bundle $\widehat{E}$ is defined on $\mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}$ minus the set S_k consisting of the points $(x, T, P) \in \mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}$ such that there is $u \in \mathrm{U}(k)$ for which

$$uTu^{-1} = \begin{pmatrix} T' & 0 \\ 0 & x \end{pmatrix}, \quad uP = \begin{pmatrix} P' \\ 0 \end{pmatrix},$$

where $(T', P') \in \mathfrak{M}^{k-1}_{\mathbb{C}}$. We are then integrating a form on the quotient of the hyperKähler reduction of $\left(\mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}\right) \setminus S_k$, i.e.

$$\mathbb{M}_k = \frac{\left((\mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}) \setminus S_k\right) /\!/\!/\!/ \mathrm{U}(k)}{\mathrm{SO}(3)}.$$

(Here SO(3) acts trivially on the manifold $\mathbb{R}^4$.)

In the infinite dimensional case, for a submanifold Σ of $\mathbb{R}^4$ we form

$$\mu_{Don}(\Sigma) = c_2(\mathbb{E})/[\Sigma] = \int_\Sigma c_2(\mathbb{E}) = \int_{\mathbb{R}^4} c_2(\mathbb{E}) \wedge \pi^*\mathrm{PD}(\Sigma_i) \in \mathrm{H}^{4-\dim\Sigma}(\mathcal{M}^*)$$

where $c_2(\mathbb{E}) \in \mathrm{H}^4(\mathbb{R}^4 \times \mathcal{M}^*)$ and $\pi : \mathbb{R}^4 \times \mathcal{M} \longrightarrow \mathbb{R}^4$ is the projection. The slant product is well-defined as an integration along the fibre because we have a perfectly decent trivial fibration $\mathbb{R}^4 \times \mathcal{M}^* \longrightarrow \mathbb{R}^4$.

However in the finite case the fibration

$$\mathbb{M}_k \longrightarrow \mathbb{R}^4$$

is destroyed because we remove points (namely S_k) from the direct product before taking the various quotients; the preimage varies topologically from point to point. If we naïvely form the finite dimensional version

$$\mu_{Don}(\Sigma) = \frac{-1}{8\pi^2}\int_\Sigma \mathrm{tr}(F(\nabla) \wedge F(\nabla))$$

then it is not altogether clear where this $\mu_{Don}(\Sigma)$ lies. The Chern-Weil representative of $\mu_{Don}(\Sigma)$ doesn't really represent a cohomology class on $\mathcal{M}$. We are forced therefore to reinterpret the situation for the ADHM case.

We have the 4-form c representing the second Chern class of the universal bundle $\widehat{E}$ over the manifold $\mathbb{M}_k$. Let $\Sigma_1, \ldots, \Sigma_k$ be compact submanifolds of $\mathbb{R}^4$ without boundary whose dimensions sum to $4l - 8k + 3$, that is of the correct dimensions to form a Donaldson polynomial. We can then consider the form

$$\mu_i = c \wedge \iota^*\pi^*\mathrm{PD}(\Sigma_i)$$

where $\pi : \mathbb{R}^4 \times \mathcal{M}_k \longrightarrow \mathbb{R}^4$ is projection, and $\iota : \mathbb{M}_k \hookrightarrow \mathbb{R}^4 \times \mathcal{M}_k$ inclusion. Thus we can form the integral

$$\int_{D_{(k,l)}} \Delta^*\left(\Pi_1^*\mu_1 \wedge \ldots \wedge \Pi_l^*\mu_l\right)$$

where $\Pi_i : \mathbb{M}_k^l \longrightarrow \mathbb{M}_k$ is projection onto the ith factor,

$$D_{(k,l)} = \{(x_1, \ldots, x_l, [T, P]) \in (\mathbb{R}^4)^l \times \mathcal{M}_k | (x_i, [T, P]) \in \mathbb{M}_k \text{ for all } k\}$$

and

$$\begin{gathered}\Delta : D_{(k,l)} \hookrightarrow (\mathbb{R}^4 \times \mathcal{M}_k)^l \\ (x_1, \ldots, x_l, [T, P]) \mapsto ((x_1, [T, P]), (x_2, [T, P]), \ldots, (x_l, [T, P]))\end{gathered}$$

a sort of diagonal map.

A similar method with the infinite dimensional case yields the construction of the Donaldson polynomial. We will go on and show that although this integral does not vanish for the $k = 1$ case where there is a discrepancy manifesting in the linking phenomena, the analogous integrals do vanish for higher k.

4 Equivariant Characteristic Classes

4.1 General Theory

We wish to find an equivariant representative of the μ-form of a submanifold in $\mathbb{R}^4$. Since this form was built up from a characteristic class it is necessary for us to consider the theory of equivariant characteristic classes.

Let $p : V \longrightarrow M$ be an equivariant G bundle over the G-manifold M. We may form the new bundle $V_G \longrightarrow M_G$ by

$$V_G = EG \times_G V \longrightarrow EG \times_G M = M_G$$

and compare the characteristic classes of V with those of V_G.

Given an equivariant connection ∇ on V we may form a connection ∇_G on V_G by pulling back by the projection $EG \times M \longrightarrow M$ and observing how it descends to the quotient. Now if K is the structure group of V, then it is also the structure group of V_G, so given a K-invariant polynomial $P \in (\bigodot \mathfrak{k}^*)^K$ we may form the characteristic classes

$$c^P = P(F(\nabla)) \in M\text{even}$$
$$c^P_G = P(F(\nabla_G)) \in \Omega^{even}_G(M)\,.$$

We should like to see how these are related.

Proposition 4.1 (Selby [10] p16)
If $f : M \longrightarrow N$ is G-equivariant between G-manifolds inducing the map

$$f_G : EG \times_G M \longrightarrow EG \times_G N$$

and $p : V \longrightarrow N$ a G-equivariant fibre bundle then

$$f^*_G V_G = (f^* V)_G$$

The proof is easy and can be found in detail in [10] Choose a base point $e \in EG$ and let $\iota : M \longrightarrow EG \times M$ be the inclusion $m \mapsto (e, m)$. So for an equivariant vector bundle $p : V \longrightarrow M$ we have

$$\begin{aligned} V &\cong \iota^*(V \times EG) \\ &\cong \iota^* q^* V_G \end{aligned}$$

where $q : EG \times M \longrightarrow M_G$ is the quotient map.We must be careful here; this is an isomorphism of bundles but is not necessarily equivariant. If we set $r = q\iota : M \longrightarrow M_G$ then $r^* : H^*_G(M) \longrightarrow H^*(M)$, and further it can also be shown that

$$r^* c^P_G = c^P$$

4.2 de Rham Theory

We now translate the equivariant theory into de Rham formalism. Recall that we make the identification

$$M_G* \longrightarrow \Omega^*_G(M) = (M* \otimes \mathbb{C}[\mathfrak{g}^*])^G$$

we can also make the identification

$$M_G; V_G* \longrightarrow (M; V* \otimes \mathbb{C}[\mathfrak{g}^*])^G$$

and call these the equivariant forms on M with values in V. Following [4], given an equivariant connection ∇ on $V \longrightarrow M$, we can form the de Rham version $\nabla_{\mathfrak{g}}$ of the corresponding connection on $V_G \longrightarrow M_G$ by setting

$$(\nabla_{\mathfrak{g}} s)(\xi) = \nabla(s(\xi)) - X_\xi \lrcorner s(\xi).$$

As mentioned in [4] pp210-211, we are motivated by

$$\mathrm{d}_{\mathfrak{g}}{}^2 s(\xi) + \mathcal{L}_\xi s(\xi) = 0$$

to define the equivariant curvature

$$F_{\mathfrak{g}}(\nabla) s(\xi) = \nabla^2_{\mathfrak{g}} s(\xi) + \mathcal{L}^V_\xi s(\xi)$$

whence

$$F_{\mathfrak{g}}(\nabla) s(\xi) = F(\nabla) s(\xi) - [\nabla, X_\xi \lrcorner] s(\xi) + \mathcal{L}_\xi s(\xi) \tag{10}$$

where $\mathcal{L}^V_\xi$ is the Lie derivative on V induced by the action of the vector field X_ξ. Now, given a K-invariant polynomial P (K being the structure group of V), we may obviously form an equivariant characteristic class

$$c^P(\xi) = P(F_{\mathfrak{g}}(\nabla)\xi \in \Omega^{\mathrm{even}}_G(M)\,.$$

How does this relate to the corresponding characteristic class of V/G on M/G?

If G does not act freely on M, then consider the manifold $M^* = M\backslash M_0$ of points with trivial stabiliser. Hence M^*/G is a manifold.

Lemma 4.2 *For a vector bundle $V \longrightarrow M^*$*

$$c^P_G(V) = q^* c^P(V/G) + \mathrm{d}_{\mathfrak{g}}\beta$$

where $q : M \longrightarrow M/G$ is the quotient map.

Proof
Given an equivariant connection ∇ on V, we may proceed as in section 5.2.3 of [6] to obtain a connection ∇' on $V/G \longrightarrow M^*/G$. In turn $q^*\nabla'$ determines an equivariant and horizontal connection on $V \longrightarrow M^*$ and hence an equivariant de Rham connection $\nabla'_{\mathfrak{g}}$ on $V_G \longrightarrow M^*_G$.

We therefore have two connections on $V_G \longrightarrow M^*_G$ namely $\nabla_{\mathfrak{g}}$ and $\nabla'_{\mathfrak{g}}$. By the usual arguments in the theory of characteristic classes

$$P(F_{\mathfrak{g}}(\nabla')) = P(F_{\mathfrak{g}}(\nabla)) + \mathrm{d}_{\mathfrak{g}}\beta.$$

But $P(F_{\mathfrak{g}}(\nabla') = q^* P(F(\nabla'))$ and defines an equivariantly closed form on M^*. So the result follows. ■

Corollary 4.3 *Any equivariant characteristic class of a bundle over a hyperKähler manifold is associated to the characteristic class of the quotient bundle over the hyperKähler reduction.*

4.3 HyperKähler Integration

We use the results developed in [7].

Definition 4.4 *Let $(M, \vec{\omega} = \omega_{\widehat{\boldsymbol{i}}}\widehat{\boldsymbol{i}} + \omega_{\widehat{\boldsymbol{j}}}\widehat{\boldsymbol{j}} + \omega_{\widehat{\boldsymbol{k}}}\widehat{\boldsymbol{k}})$ be a hyperKähler manifold which possesses a tri-Hamiltonian action of the compact Lie group G and $\alpha \in \Omega^{\bullet}_G(M)$ be an equivariant form. We shall say that α is associated to $\alpha_0 \in \mathcal{M}\bullet$ if*

$$\iota^*\alpha = \pi^*\alpha_0 + \mathrm{d}_{\mathfrak{g}}\beta$$

where $\iota : \vec{\mu}^{-1}(0) \longrightarrow M$ is inclusion and $\pi : \vec{\mu}^{-1}(0) \longrightarrow \mathcal{M}$ is the quotient map and $\beta \in \Omega^{\bullet}_G(\vec{\mu}^{-1}(0))$. In the case that α and α_0 are compactly supported, we shall say that α is compactly associated to α_0 if α is associated to α_0 as above, and the form β is also compactly supported.

The reduction of $(M, \vec{\omega})$ is a manifold $(M /\!/\!/\!/ G, \vec{\omega}_0)$. The compactness of support is a necessity from the fact communicated to the Author by Roger Bielawski that there are no compact hyperKähler reductions by a tri-Hamiltonian group. One of the main results of [7] (also found in [8]) is

Theorem 4.5 *If η is compactly associated to η_0, then*

$$\begin{aligned}
&\int_{\mathcal{M}} e^{i\vec{\omega}_0 \curlywedge \vec{\omega}_0} \left(4i\vec{\omega}_0 \curlywedge \vec{\omega}_0 + 1\right) \eta_0 \\
= &\left(\frac{1}{6\pi i\sqrt{2}}\right)^k \frac{1}{|W|} \oint_M e^{i\vec{\omega}\curlywedge\vec{\omega} + i|\vec{\mu}|^2 y} \mathrm{Pr}_{ev}\left(z \mapsto e^{2iz\vec{\mu}.\vec{\omega}} w(z)^4 \eta(z)\right)(\sqrt{y})
\end{aligned}$$

where

$$\oint_M \alpha(\xi) = \frac{1}{\mathrm{vol}\,(G)} \lim_{t \longrightarrow \infty} \int_{\mathfrak{g}} \mathrm{d}\xi e^{-\frac{|\xi|^2}{4t}} \int_M \alpha(\xi),$$

and

$$\vec{\omega} \curlywedge \vec{\omega} = \omega_{\hat{\boldsymbol{i}}} \wedge \omega_{\hat{\boldsymbol{i}}} + \omega_{\hat{\boldsymbol{j}}} \wedge \omega_{\hat{\boldsymbol{j}}} + \omega_{\hat{\boldsymbol{k}}} \wedge \omega_{\hat{\boldsymbol{k}}},$$

the operator $\mathrm{Pr}_{ev} : \Omega^\bullet_G(M) \longrightarrow \Omega^\bullet_G(M)$ *is*

$$\left(\mathrm{Pr}_{ev}(\beta)\right)(z) = \frac{1}{2}\left(\beta(z) + \beta(-z)\right),$$

$|W|$ *is the number of elements of the Weyl Group associated with the maximal torus in* G *and*

$$w(y) = \prod_{\alpha \in \Delta_+} \alpha(y)$$

is the polynomial formed by the product of the positive roots of G.

It is this result that allows us to compute the integrals in which we are primarily interested.

5 Applications to the ADHM construction

We wish to apply the results on equivariant integration and localisation for hyperKähler quotients to the ADHM construction. Here it is better to pass to the "complex" version given by

$$(T, P) \in \mathfrak{M}^k_{\mathbb{C}} = \left(i\mathfrak{u}(k) \oplus \mathbb{C}^k\right) \otimes_{\mathbb{R}} \mathbb{H}$$

which has the moment map

$$\vec{\mu}(T,P) = \Im_{\mathbb{H}}(T^*T + PP^*)$$

where $\Im_{\mathbb{H}}$ is the complexification of the operation of taking the quaternionic imaginary part and $*$ means taking the quaternionic conjugate of the complex adjoint.

The reason for this change of approach is that

$$\widetilde{\mathcal{M}} = \mathfrak{M}^k_{\mathbb{R}} /\!\!/\!\!/\!\!/ \text{Ø}k \cong \mathfrak{M}^k_{\mathbb{C}} /\!\!/\!\!/\!\!/ \mathrm{U}(k)$$

and $\mathrm{U}(k)$ is connected and has a simpler Lie algebra structure than $\text{Ø}k$. We may also describe the maximal torus of $\mathrm{U}(k)$ more simply than $\text{Ø}k$ and we will be using this to assist us in our localisation. This does not affect any of our previous results.

5.1 The Equivariant Euler Class

Our first priority is to work out the fixed point set of the action of the maximal torus

$$\mathbb{T}^k \subset \mathrm{U}(k).$$

To do this, we take the decomposition

$$\mathbb{T}^k = \mathbb{T}_1 \times \ldots \times \mathbb{T}_k$$

where

$$\mathbb{T}_j = \left\{ \left(\begin{array}{ccc} \mathbb{1} & 0 & 0 \\ 0 & e^{i\theta_j} & 0 \\ 0 & 0 & \mathbb{1} \end{array} \right) \middle| \theta_j \in \mathbb{R} \right\}.$$

By finding the fixed set of $\mathbb{T}_i$ and the equivariant Euler class of its normal bundle in $\mathfrak{M}^k_{\mathbb{C}}$ we will be able to apply our inductive formula.

Let $\xi_j \in \mathbb{T}_i$ be the generator

$$\xi_j = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{array} \right).$$

Then the vector field X_{ξ_j} on $\mathfrak{M}^k_{\mathbb{C}}$ is given by

$$\begin{aligned} X_{\xi_j}(T,P) &= ([\xi_j, T], \xi_j P) \\ &= \left(i \begin{bmatrix} 0 & -T_1 & 0 \\ T_1^* & 0 & T_4 \\ 0 & -T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ iP_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right), \end{aligned}$$

where

$$T = \begin{pmatrix} T' & T_1 & T_2 \\ T_1^* & T'' & T_4 \\ T_2^* & T_4^* & T''' \end{pmatrix}, \text{ and } P = \begin{pmatrix} P_1 \\ \vdots \\ P_k \end{pmatrix}.$$

Let Sing_j^k consist of the points where X_{ξ_j} vanishes, i.e

$$\mathrm{Sing}_j^k := \left\{ \left(\begin{bmatrix} T' & 0 & T_2 \\ 0 & T'' & 0 \\ T_2^* & 0 & T''' \end{bmatrix}, \begin{bmatrix} P' \\ 0 \\ P'' \end{bmatrix} \right) \in \mathfrak{M}_{\mathbb{C}}^k \right\}.$$

Theorem 5.1 *The* S^1*-equivariant Euler class* e *of the normal bundle of* Sing_j^k *in* $\mathfrak{M}_{\mathbb{C}}^k$ *is given by*

$$e(\lambda) = \left(\frac{\lambda}{2\pi}\right)^{4k}.$$

Proof

Set $\mathcal{V}_j = \mathcal{V}(\mathrm{Sing}_j^k \hookrightarrow \mathfrak{M}_{\mathbb{C}}^k)$, the normal bundle of Sing_j^k in $\mathfrak{M}_{\mathbb{C}}^k$. We notice that Sing_j^k is a vector subspace of $\mathfrak{M}_{\mathbb{C}}^k$, so

$$\begin{aligned} \mathrm{T}_{(T,P)}\mathrm{Sing}_j^k &\cong \mathrm{Sing}_j^k, \\ \mathrm{T}_{(T,P)}\mathfrak{M}_{\mathbb{C}}^k &\cong \mathfrak{M}_{\mathbb{C}}^k, \end{aligned}$$

canonically. So

$$(\mathcal{V}_j)_{(T,P)} = \left(\mathrm{T}_{(T,P)}\mathrm{Sing}_j^k\right)^{\perp} = (\mathrm{Sing}_j^k)^{\perp}.$$

It can be shown that

$$\begin{aligned} (\mathrm{Sing}_j^k)^{\perp} &= \{X_{\xi_j}(T,P) | (T,P) \in \mathfrak{M}_{\mathbb{C}}^k\} \\ &= \left\{ \left(\begin{bmatrix} 0 & T_1 & 0 \\ -T_1^* & 0 & T_4 \\ 0 & -T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ P_j \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) \middle| \begin{array}{ccc} T_1 & \in & \mathbb{C}^{j-1} \otimes_{\mathbb{R}} \mathbb{H}, \\ T_4 & \in & \mathbb{C}^{k-j} \otimes_{\mathbb{R}} \mathbb{H}, \\ P_j & \in & \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \end{array} \right\}. \end{aligned}$$

Since $\mathcal{V}_j$ is an equivariant bundle, and the de-Rham operator is a perfectly good equivariant connection on $\mathcal{V}_j$, we see automatically from (10) that the equivariant curvature

$$F_{\mathfrak{g}}(\mathrm{d})(\lambda\xi_j) = \mathcal{L}_{\lambda\xi_j}^{\mathcal{V}_j}.$$

Now,

$$\mathfrak{L}_{\lambda\xi_j}^{\mathcal{V}_j}\left(\begin{bmatrix} 0 & T_1 & 0 \\ -T_1^* & 0 & T_4 \\ 0 & -T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ P_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \lambda i \left(\begin{bmatrix} 0 & -T_1 & 0 \\ T_1^* & 0 & -T_4 \\ 0 & T_4^* & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ P_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$$

More simply

$$\mathfrak{L}_{\lambda\xi_j}^{\mathcal{V}_j}(T_1, T_4, P_j) = (-i\lambda T_1, -i\lambda T_4, i\lambda P_j).$$

From this we are able to deduce that

$$\begin{aligned} e_j(\lambda) &= \operatorname{Pfaff}\left(\frac{1}{2\pi} F_{\mathfrak{g}}(\lambda\xi_j)\right) \\ &= \left(\frac{-\lambda}{2\pi}\right)^{4(k-1)} \left(\frac{\lambda}{2\pi}\right)^{4} \\ &= \left(\frac{\lambda}{2\pi}\right)^{4k}. \end{aligned}$$

■

5.2 The Universal 2nd Chern Class

In order to find the Universal equivariant 2nd Chern class, we apply the theory in 4.2 to the connection $\widehat{\nabla} = v^* \circ \widehat{\mathrm{d}} \circ v$ on the the bundle $\widehat{E} \longrightarrow \mathbb{R}^4 \times \mathfrak{M}_{\mathbb{C}}^k$ which has curvature

$$F(\widehat{\nabla}) = v^* \widehat{\mathrm{d}}\mathfrak{R}^* \wedge \mathfrak{F}\widehat{\mathrm{d}}\mathfrak{R}v.$$

Using this it can be quite easily shown that for some section local section $s \in \mathfrak{M}_{\mathbb{C}}^k; \widehat{E}0$ and $\xi \in \mathfrak{u}(k)$, we have

$$\begin{aligned} \widehat{\nabla}_{X_\xi} s &= X_\xi \lrcorner v^* \widehat{\mathrm{d}} v s \\ &= v^* \widehat{\mathrm{d}}(vs)(X_\xi) \\ &= v^* \frac{\partial}{\partial t}\bigg|_{t=0} \left(\phi_t^{(\xi)}\right)^* vs \\ &= v^* \left(\frac{\partial}{\partial t}\bigg|_{t=0} \left(\phi_t^{(\xi)}\right)^* v \right) s + \frac{\partial}{\partial t}\bigg|_{t=0} \left(\phi_t^{\xi}\right)^* s \\ &= v^* \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} vs + \mathfrak{L}_\xi s. \end{aligned}$$

Hence

$$\begin{aligned} F_{\mathfrak{g}}(\widehat{\nabla})(\xi) &= F(\widehat{\nabla}) - v^* \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} v \\ &= v^* \left(\widehat{\mathrm{d}}\mathcal{R}^* \wedge \mathcal{F}\widehat{\mathrm{d}}\mathcal{R} - \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \right) v \end{aligned}$$

and obviously our representative of the equivariant universal Chern class will be

$$c^k(\xi) = \frac{1}{4\pi^2} \mathrm{tr} \left(F_{\mathfrak{g}}(\widehat{\nabla})(\xi) \wedge F_{\mathfrak{g}}(\widehat{\nabla})(\xi) \right) \in \Omega^4_{\mathbb{T}^k}(\mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}).$$

However, c^k is integrable but not smooth on the space of reducibles. To understand c^k on the space of reducibles, if we look at $c^k_{(x,T,P)}$ for

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 \\ P_1 \end{pmatrix}$$

where $T_0 \in \mathbb{H}$, $(T_1, P_1) \in \mathfrak{M}^{k-1}_{\mathbb{C}}$ and $x \neq T_0$, then it is straightforward to show that

$$c^k_{(x,T,P)}(\xi) = c^{k-1}_{(x,T_1,P_1)}(\xi_1)$$

where ξ_1 is the element of $\mathrm{Lie}\mathbb{T}^{k-1}$ got from ξ by removing the first row and column. But,

$$\int_{\mathbb{R}^4} c^k_{(x,t,p)}(\xi) \mathrm{d}x = k$$

for all irreducible $(t, p) \in \mathfrak{M}^k_{\mathbb{C}}$.

Lemma 5.2 *For the above* (T, P)

$$\iota_1^* c^k_{(x,T,P)}(\xi) = c^{k-1}_{(x,T_1,P_1)}(\xi_1) + \delta(x-T_0)\mathrm{d}(x-T_0)_1 \wedge \mathrm{d}(x-T_0)_2 \wedge \mathrm{d}(x-T_0)_3 \wedge \mathrm{d}(x-T_0)_4.$$

where $\iota_1 : \mathrm{Sing}^k_1 \hookrightarrow \mathfrak{M}^k_{\mathbb{C}}$ *is inclusion and if*

$$\xi = \begin{pmatrix} \xi_{11} & 0 & & \\ 0 & \xi_{22} & & \\ & & \ddots & \\ & & & \xi_{kk} \end{pmatrix}$$

we let

$$\xi_1 = \begin{pmatrix} \xi_{22} & & \\ & \ddots & \\ & & \xi_{kk} \end{pmatrix}.$$

Proof
Set

$$L(\xi) = \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix}.$$

The splitting of the matrix $L(\xi)$ is in terms of the splitting of $\mathbb{C}^k \otimes_{\mathbb{R}} \mathbb{H} \oplus \mathbb{C}^2$. We have another splitting to consider here due to separating T into T_0 and T_1. Let

$$L(\xi) = \begin{pmatrix} \xi_{11} & 0 \\ 0 & L_1(\xi_1) \end{pmatrix},$$

where $L_1(\xi_1)$ plays same rôle as L for $\mathfrak{M}_{\mathbb{C}}^{k-1}$.

We have

$$\mathcal{R}_{(x,T,P)} = \begin{pmatrix} (T_0 - x)^* & 0 & 0 \\ 0 & (T_1 - x\mathbb{1})^* & P_1 \end{pmatrix} = \begin{pmatrix} -\tilde{x}_0^* & 0 \\ 0 & \mathcal{R}_1 \end{pmatrix}$$

where $\tilde{x_0} = x - T_0$ and $\mathcal{R}_1 = \mathcal{R}_{(x,T_1,P_1)}$. Now

$$\mathcal{F} = (\mathcal{R}\mathcal{R}^*)^{-1} = \begin{pmatrix} |\tilde{x}_0|^{-2} & 0 \\ 0 & \mathcal{F}_1 \end{pmatrix}$$

where $\mathcal{F}_1 = (\mathcal{R}_1\mathcal{R}_1^*)^{-1}$. Since there is a singularity when $x = T_0$, we make a small adjustment depending on a parameter ρ which we will shrink to 0.

Set

$$\mathcal{F}_\rho = \begin{pmatrix} \frac{1}{(\rho^2 + |\tilde{x}_0|^2)} & 0 \\ 0 & \mathcal{F}_1 \end{pmatrix}.$$

Define

$$\varpi_\rho = \mathbb{1} - \mathcal{R}^*\mathcal{F}_\rho\mathcal{R} = \begin{pmatrix} \frac{\rho^2}{(\rho^2 + |\tilde{x}_0|^2)} & 0 \\ 0 & \varpi_1 \end{pmatrix}$$

where $\varpi_1 = \mathbb{1} - \mathcal{R}_1^*\mathcal{F}_1\mathcal{R}_1$. We note that

$$\iota_1^* \mathrm{d}\mathcal{R} = \begin{pmatrix} -\widehat{\mathrm{d}}\tilde{x}_0^* & 0 \\ 0 & \widehat{\mathrm{d}}\mathcal{R}_1 \end{pmatrix}.$$

So we have

$$
\begin{aligned}
& \iota_1^* c^k_{(x,T,P)}(\xi) \\
= \;& \lim_{\rho \longrightarrow 0} \frac{1}{4\pi^2} \Re \mathrm{tr} \left[\left((\iota_1^* \widehat{\mathrm{d}} \mathcal{R}^* \wedge \mathcal{F}_\rho \iota_1^* \widehat{\mathrm{d}} \mathcal{R} \varpi - L(\xi)\varpi \right)^2 \right] \\
= \;& \lim_{\rho \longrightarrow 0} \frac{1}{4\pi^2} \Re \left(\frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^4} \widehat{\mathrm{d}}\tilde{x}_0 \wedge \widehat{\mathrm{d}}\tilde{x}_0^* \wedge \widehat{\mathrm{d}}\tilde{x}_0 \wedge \widehat{\mathrm{d}}\tilde{x}_0^* \right. \\
- \;& \left. \frac{2\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^3} \widehat{\mathrm{d}}\tilde{x}_0 \wedge \widehat{\mathrm{d}}\tilde{x}_0^* \xi_{11} + \frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)} \xi_{11}^2 \right) \\
+ \;& c^{k-1}_{(x,T_1,P_1)}(\xi_1) \\
= \;& \lim_{\rho \longrightarrow 0} \frac{1}{4\pi^2} \Re \left(\frac{24\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^4} \widehat{\mathrm{d}}(\tilde{x}_0)_1 \wedge \widehat{\mathrm{d}}(\tilde{x}_0)_2 \wedge \widehat{\mathrm{d}}(\tilde{x}_0)_3 \wedge \widehat{\mathrm{d}}(\tilde{x}_0)_4 \right. \\
- \;& \left. \frac{2\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^3} \widehat{\mathrm{d}}\tilde{x}_0 \wedge \widehat{\mathrm{d}}\tilde{x}_0^* \xi_{11} + \frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)} \xi_{11}^2 \right) \\
+ \;& c^{k-1}_{(x,T_1,P_1)}(\xi_1).
\end{aligned}
$$

Now the terms of the form

$$\frac{\rho^4}{(\rho^2 + |\tilde{x}_0|^2)^n}$$

are distributional in the limit as $\rho_{ii} \longrightarrow 0$, so it is worth integrating them against a compactly supported test function $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$ that is, calculating

$$\lim_{\rho \longrightarrow 0} \int_{\mathbb{R}^4} \frac{\rho^4}{(\rho^2 + |y|^2)^n} f(y) \mathrm{d}y.$$

It isn't hard to show that as distributions

$$\lim_{\rho \longrightarrow 0} \frac{\rho^4}{(\rho^2 + |y|^2)^4} = \frac{\pi^2}{6} \delta(y),$$
$$\lim_{\rho \longrightarrow 0} \frac{\rho^4}{(\rho^2 + |y|^2)^3} = 0$$

and after a little work is is possible to show that

$$\lim_{\rho \longrightarrow 0} \frac{\rho^4}{(\rho^2 + |y|^2)^2} = 0$$

as a distribution. We are therefore left with the result that

$$\iota_1^* c^k_{(x,T,P)}(\xi) = c^{k-1}_{(x,T_1,P_1)}(\xi_1) + \delta(x - T_0) \mathrm{d}(x - T_0)_1 \wedge \mathrm{d}(x - T_0)_2 \wedge \mathrm{d}(x - T_0)_3 \wedge \mathrm{d}(x - T_0)_4.$$

as stated. $\blacksquare$

For a submanifold $\Sigma \subset \mathbb{R}^4$, define

$$\mu^k(\Sigma)_{(T,P)}(\xi) = \int_{x\in\Sigma} c^k_{(x,T,P)}(\xi) = \int_{x\in\mathbb{R}^4} c^k_{(x,T,P)}(\xi) \wedge PD(\Sigma)_x.$$

Corollary 5.3 *With T, P, ι_1 as above, we have*

$$\iota_1^*\mu^k(\Sigma)_{(x,T,P)}(\xi) = \mu^{k-l}(\Sigma)_{(T_1,P_1)}(\xi_1) + \mathrm{PD}(\Sigma)_{T_0}.$$

Corollary 5.4 *If ι is the inclusion of the set of points of $\mathfrak{M}^k_{\mathbb{C}}$ fixed under $\mathbb{T}^k$, and $p_i : \mathfrak{M}^k_{\mathbb{C}} \longrightarrow \mathbb{R}^4$ is the projection $T \mapsto T_{ii}$ then*

$$\iota^*\mu^k(\Sigma)(\xi) = \sum_{i=1}^k p_i^*\mathrm{PD}(\Sigma).$$

Corollary 5.5 *If $\mu^k(\Sigma) \in \frac{\mathfrak{M}^k_{\mathbb{C}}}{\mathrm{U}(k)\mathrm{Sp}(1)} 4 - \dim\sigma$ then for (T, P) and ι_1 as above we have*

$$\iota_1^*\pi^*p^*\mu^k(\Sigma)_{(T,P)} = \pi^*p^*\mu^{k-1}(\Sigma)_{(T_1,P_1)} + p_1^*\mathrm{PD}(\Sigma)_{T_0} + \pi^*p^*\widehat{\mathrm{d}}\gamma(\Sigma).$$

for some $\gamma(\Sigma) \in \frac{\mathfrak{M}^k_{\mathbb{C}}}{\mathrm{U}(k)\mathrm{Sp}(1)} 3 - \dim\Sigma$, and p_1 the projection as in Corollary 5.4.

Proof
Now, we have a chain of quotients

$$\begin{array}{ccccc} \mathfrak{M}^k_{\mathbb{C}} & \stackrel{\pi}{\longrightarrow} & \frac{\mathfrak{M}^k_{\mathbb{C}}}{\mathrm{U}(k)} & \stackrel{p}{\longrightarrow} & \frac{\mathfrak{M}^k_{\mathbb{C}}}{\mathrm{U}(k)\mathrm{Sp}(1)} \\ & & \cup & & \cup \\ & & \widetilde{\mathcal{M}}_k & & \mathcal{M}_k \end{array}.$$

So the form $\pi^*p^*\mu^k(\Sigma)$ is now a closed, $\mathrm{U}(k)$-basic $4 - \dim\Sigma$ degree form on $\mathfrak{M}^k_{\mathbb{C}}$. Since $\mu^k(\Sigma)(\xi) \in \Omega_{\mathrm{U}(k)}(\mathfrak{M}^k_{\mathbb{C}})$ was formed using equivariant Chern-Weil theory and by Lemma 4.2, we have

$$\mu^k(\Sigma)(\xi) = \pi^*p^*\mu^k(\Sigma) + \int_\Sigma \mathrm{d}_{\mathfrak{g}}\beta^k(\xi).$$

Also by Corollary 5.3 we have

$$\iota_1^*\mu^k(\Sigma)_{(x,T,P)}(\xi) = \mu^{k-l}(\Sigma)_{(T_1,P_1)}(\xi_1) + \mathrm{PD}(\Sigma)_{T_0}.$$

Thus

$$\iota_1^*\pi^*p^*\mu^k(\Sigma) = \pi^*p^*\mu^{k-1}(\Sigma)_{(T_1,P_1)} + \mathrm{PD}(\Sigma)_{T_0} + \int_\Sigma \mathrm{d}_{\mathfrak{g}}\left(\beta^{k-1} - \iota_1^*\beta^k\right)(\xi).$$

The left hand side is independent of ξ, so $\mathrm{d}_{\mathfrak{g}}\left(\beta^{k-1} - \iota_1^*\beta^k\right)(\xi)$ is an exact, $\mathrm{U}(k)$-basic form, thus the right hand side is in the same de Rham cohomology class as the left. The result follows since $\pi^*p^*\mu^k(\Sigma)$, $\pi^*p^*\mu^{k-1}(\Sigma)$ and the Poincaré dual are $\mathrm{U}(k)\mathrm{Sp}(1)$-basic. By construction, $\iota_1^*\mathrm{d}_{\mathfrak{g}}\beta^k$ does not depend on T_0 and thus agrees with $\mathrm{d}_{\mathfrak{g}}\beta^{k-1}$ giving the result. ■

5.3 Integrability of the Donaldson μ map

What is not altogether clear is that the form representing the Donaldson polynomial is actually integrable. Indeed there are various technicalities in forming the these polynomials that Donaldson and Kronheimer discuss in Chapter 9 of [6]. Our approach will be from a functional analytic viewpoint.

Definition 5.6 *Given pairwise disjoint, compact submanifolds $\Sigma_1, \ldots \Sigma_l$ of $\mathbb{R}^4$ we define the Donaldson functional on compactly supported functions of $\frac{\mathfrak{M}_{\mathbb{C}}^k}{\mathrm{U}(k)\mathrm{Sp}(1)}$ by*

$$\mathrm{Don}_k(\Sigma_1, \ldots, \Sigma_l)(\phi) = \int_{\mathcal{M}_k} \phi\, \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l).$$

This is certainly well defined; the above argument shows that the representatives of the μ classes on the reducible space are distributional in nature and integrable, and thus the integral exists for any compactly supported function ϕ.

5.4 Computing the integrals

We must, however, make a slight alteration to the situation since the action of $\mathrm{U}(k)$ is not free on $\vec{\mu}^{-1}(0)$. Instead we choose $\vec{\zeta}_0 \in \Im\mathbb{H}$ and take the moment map to be

$$\vec{\mu}_{\zeta_0}(T, P) = \Im(T^*T + PP^*) - \vec{\zeta}_0 \mathbb{1}.$$

We have to decide on how best to approach the integration.

Let $\Sigma_1, \ldots, \Sigma_l$ be pairwise disjoint, compact submanifolds of $\mathbb{R}^4$ of dimensions $d_1, \ldots, d_l$ respectively such that

$$\sum_{i=1}^{l}(4 - d_i) = 8k - 3,$$

that is

$$\sum_{i=1}^{l} d_i = 4l - 8k + 3.$$

Then $\alpha = \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l)$ is represented by a form of top degree on $\mathcal{M}_k$.

Now the de-Rham operator $\widehat{\mathrm{d}}$ splits on $\mathbb{R}^4 \times \mathcal{M}_k$

$$\widehat{\mathrm{d}} = \mathrm{d} + \delta$$

where δ is the de Rham differential on $\mathcal{M}$. Since both $\mathrm{U}(k)$ and $\mathrm{Sp}(1)$ act trivially on $\mathbb{R}^4$, we see that

$$\mathrm{d}_{\mathfrak{g}}\eta(\gamma) = \mathrm{d}\eta(\gamma) + \delta\eta(\gamma) - X_\gamma \lrcorner \eta(\gamma) = \widehat{\mathrm{d}}\eta(\gamma) - X_\gamma \lrcorner \eta(\gamma),$$

for any $\eta \in \Omega^\bullet_G \left(\mathbb{R}^4 \times \mathfrak{M}^k_{\mathbb{C}}\right)$, where following the notation in Chapter 2, we reserve d for the de-Rham differential on $\mathbb{R}^4$ and δ the differential on $\mathfrak{M}^k_{\mathbb{C}}$ and the total differential $\widehat{\mathrm{d}} = \mathrm{d} + \delta$. Now suppose without loss of generality that Σ_1 is not a point, and that Ξ_1 is a Seifert surface spanning Σ_1, i.e $\partial \Xi_1 = \Sigma_1$. Then

$$\begin{aligned} 0 &= \int_{\Xi_1} \widehat{\mathrm{d}} c^k \\ &= \int_{\Xi_1} \mathrm{d} c^k + \int_{\Xi_1} \delta c^k \\ &= \int_{\Sigma_1} c^k + \delta \int_{\Xi_1} c^k \end{aligned}$$

i.e

$$\int_{\Sigma_1} c^k = -\delta \int_{\Xi_1} c^k. \tag{11}$$

Hence we may take

$$\beta = -\int_{\Xi_1} c^k \wedge \int_{\Sigma_2} c^k \wedge \ldots \wedge \int_{\Sigma_l} c^k$$

and

$$\alpha = \delta\beta.$$

So as we saw above,

$$\begin{aligned} & \mathrm{Don}_k(\Sigma_1, \ldots, \Sigma_l)(\phi) \\ = & \int_{\mathcal{M}_k} \phi\, \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) \\ = & \int_{\widetilde{\mathcal{M}}_k} p^*\phi\, p^*\widehat{\mathrm{d}} \left(\mu(\Xi_1) \wedge \ldots \wedge \mu(\Sigma_l)\right) \wedge \Theta \\ = & \int_{\widetilde{\mathcal{M}}_k} e^{i\vec{\omega}_0 \wedge \vec{\omega}_0} \left(4i\vec{\omega}_0 \wedge \vec{\omega}_0 + 1\right) p^*\phi\, p^*\mathrm{d} \left(\mu(\Xi_1) \wedge \ldots \wedge \mu(\Sigma_l)\right) \wedge \Theta \end{aligned}$$

since

$$p^*\phi p^*\mathrm{d}\left(\mu(\Xi_1)\wedge\ldots\wedge\mu(\Sigma_l)\right)\wedge\Theta$$

already has maximal degree. This form is associated to

$$\eta=\pi^*\left(p^*\phi p^*\widehat{\mathrm{d}}\left(\mu(\Xi_1)\wedge\ldots\wedge\mu(\Sigma_l)\right)\wedge\Theta\right)\in\Omega^{8k}_{\mathrm{U}(k)}\left(\mathfrak{M}^k_{\mathbb{C}}\right),$$

which is basic and de Rham closed by construction and compactly supported.

We can now use Theorem 4.5 applied to this form.

$$\begin{aligned}
&\mathrm{Don}_k(\Sigma_1,\ldots,\Sigma_l)(\phi)\\
=&\left(\frac{1}{6\pi i\sqrt{2}}\right)^k\frac{1}{|S_k|}\oint_{\mathfrak{M}^k_{\mathbb{C}}}e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}_{\zeta_0}|^2y}\mathrm{Pr}_{ev}\left(z\mapsto e^{2iz\vec{\mu}_{\zeta_0}.\vec{\omega}}w(z)^4\eta\right)(\sqrt{y}),\\
=&\left(\frac{1}{6\pi i\sqrt{2}}\right)^k\frac{1}{k!}\oint_{\mathfrak{M}^k_{\mathbb{C}}}e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}_{\zeta_0}|^2y}\mathrm{Pr}_{ev}\left(z\mapsto e^{2iz\vec{\mu}_{\zeta_0}.\vec{\omega}}w(z)^4\right)(\sqrt{y})\eta,
\end{aligned}$$

and use the localisation theorem to prove

Theorem 5.7

$$\mathrm{Don}_k(\Sigma_1,\ldots,\Sigma_l)(\phi)=\lambda(|\vec{\zeta}_0|)\mathcal{P}(\Sigma_1,\ldots,\Sigma_l)(\phi)$$

for λ a suitable polynomial, and $\mathcal{P}$ a topological quantity depending on the arrangements of the Σ_i in $\mathbb{R}^4$ and upon the test function ϕ .

Proof

We localise the integral with respect to the $(k-1)$-torus in stead of the k-torus since there is a problem with the form Θ at $|P|=0$ which is the fixed set of the full k-torus . We hope to be able to express the integral then in terms of the Donaldson polynomials for charge $k=1$. We are not interested in the constant multiples that occur here, so they will be largely forgotten.

$$\begin{aligned}
&\mathrm{Don}_k(\Sigma_1,\ldots,\Sigma_l)(\phi)\\
=&\left(\frac{1}{6\pi i\sqrt{2}}\right)^k\frac{1}{k!}\oint_{\mathfrak{M}^k_{\mathbb{C}}}e^{i\vec{\omega}\wedge\vec{\omega}+i|\vec{\mu}_{\zeta_0}|^2y}\mathrm{Pr}_{ev}\left(z\mapsto e^{2iz\vec{\mu}_{\zeta_0}.\vec{\omega}}w(z)^4\right)(\sqrt{y})\eta,\\
=&\ const\oint_{(\mathbb{R}^4)^{k-1}\times\mathfrak{M}^1_{\mathbb{C}}}\mathrm{Coeff}_{y_1^{-2}\ldots y_{k-1}^{-2}}\left[\frac{\iota^*e^{i\vec{\omega}\wedge\vec{\omega}+i\sum_{\nu=1}^k\left(|\vec{\zeta}_0|^2y_\nu^2+2y_\nu\vec{\zeta}_0.\vec{\omega}\right)}w(y)^4}{y_1^{4k}y_2^{4k-4}\ldots y_{k-1}^8}\right]\iota^*\eta.
\end{aligned}$$

Now restricted to the fixed set of the $(k-1)$-torus we have by Corollary 5.5

$$\iota^*\eta\ =\ \iota^*\prod_{j=1}^l\left(\pi^*p^*\mu^1(\Sigma_j)+\sum_{m=1}^{k-1}p_i^*PD(\Sigma_j)\right)\wedge\iota^*\pi^*\Theta$$

where p_i are the projections described in Corollary 5.4. Thus the form of the Donaldson polynomial can be seen certainly as the product of a polynomial in $|\vec{\zeta}_0|$ and a sum of integrals of the form

$$\begin{aligned} &\int_{\mathfrak{M}^1_{\mathbb{C}}\times(\mathbb{R}^4)^{k-1}} (\iota^*\pi^*p^*\phi)\ \pi^*\left(\Theta\wedge\prod_{i\in I}p^*\mu^1(\Sigma_i)\right)\wedge\prod_B\prod_{b\in B}\mathrm{PD}(\Sigma_b) \\ = &\int_{\mathcal{M}_1\times(\mathbb{R}^4)^{k-1}} (\iota^*\phi)\left(\prod_{i\in I}\mu^1(\Sigma_i))\right)\wedge\prod_B\prod_{b\in B}\mathrm{PD}(\Sigma_b) \end{aligned}$$

where I is a subset of $\{1,\ldots,l\}$, and B runs over all subsets that form a partition of $\{1,\ldots,l\}\backslash I$. ■

5.5 $k=1$ Revisited from the Topological Viewpoint

We present a slightly different approach to the theory of $k=1$. Here we use the Poincaré duality property detailed in Donaldson's paper [5]. We state it in the version it appears in [6].

Lemma 5.8 (Corollary 5.3.3 of [6] p199) *Let X be a simply connected Riemannian 4-manifold and $E\longrightarrow X$ have $c_2(E)=1$, and let $\tau : X\longrightarrow \mathcal{B}^*_{X,E}$ be any map into the space of the gauge equivalence classes of irreducible connections on E with the property that for all x, the connection $\tau(x)$ is flat and trivial outside some ball of finite diameter centred on x. Then the composite*

$$\mathrm{H}_2(X;\mathbb{Z})\xrightarrow{\mu}\mathrm{H}^2(\mathcal{B}^*_{X,E})\xrightarrow{\tau^*}\mathrm{H}^2(X;\mathbb{Z})$$

is the Poincaré duality isomorphism.

Definition 5.9 *We will call such a τ a tractator*

This can be proved at the level of forms to show that in the case of $X=\mathrm{S}^4$ we have the following

Lemma 5.10 *For a tractator $\tau:\mathrm{S}^4\longrightarrow\mathcal{B}^*_{\mathrm{S}^4,E}$, we have for each submanifold Σ of S^4*

$$\int_\Sigma \iota^*_\Sigma\alpha=\int_{\mathrm{S}^4}\alpha\wedge\tau^*\mu(\Sigma)$$

for any $\alpha\in\mathrm{S}^4\dim\Sigma$, where ι_Σ is the inclusion of Σ in S^4.

Now, for $\varepsilon > 0$ let

$$\mathfrak{M}^1_{\mathbb{C}\varepsilon} = \{ (T,P) \in \mathfrak{M}^1_{\mathbb{C}} \big| \, |P| \geq \varepsilon \}$$

and

$$\mathcal{M}_\varepsilon = (\mathfrak{M}^1_{\mathbb{C}\varepsilon} /\!\!/\!\!/\!\!/ \mathrm{U}(1))/\mathrm{Sp}(1).$$

This is a manifold with boundary.

Let $\Sigma_1, \ldots, \Sigma_l$ be pairwise disjoint, compact submanifolds of S^4 with dimensions $d_1, \ldots, d_l$ respectively such that

$$d_1 + \ldots + d_l = 4l - 5.$$

Suppose w.l.o.g that Σ_1 is not a point and let Ξ_1 be a Seifert manifold for it. then from above we know that

$$\mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) = \mathrm{d}\left(\mu(\Xi_1) \wedge \mu(\Sigma_2) \wedge \ldots \wedge \mu(\Sigma_l)\right).$$

Hence

$$\begin{aligned}
\int_{\mathcal{M}} \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) &= \lim_{\varepsilon \longrightarrow 0} \int_{\mathcal{M}_\varepsilon} \mu(\Sigma_1) \wedge \ldots \wedge \mu(\Sigma_l) \\
&= \lim_{\varepsilon \longrightarrow 0} \int_{\partial \mathcal{M}_\varepsilon} \mu(\Xi_1) \wedge \mu(\Sigma_2) \wedge \ldots \wedge \mu(\Sigma_l) \\
&= \int_{\mathrm{S}^4} \tau^* \mu(\Xi_1) \wedge \tau^* \mu(\Sigma_2) \wedge \ldots \wedge \tau^* \mu(\Sigma_l) \\
&\quad \text{for some appropriate tractator } \tau. \\
&= \int_{\mathrm{S}^4} \mathrm{PD}(\Xi_1) \wedge \mathrm{PD}(\Sigma_2) \wedge \ldots \wedge \mathrm{PD}(\Sigma_l) \\
&= \text{intersection number of } \bigcap_{i=2}^{l} \Sigma_i \text{ with } \Xi_1 \\
&= \text{linking number of } \Sigma_1 \text{ and } \bigcap_{i=2}^{l} \Sigma_i
\end{aligned}$$

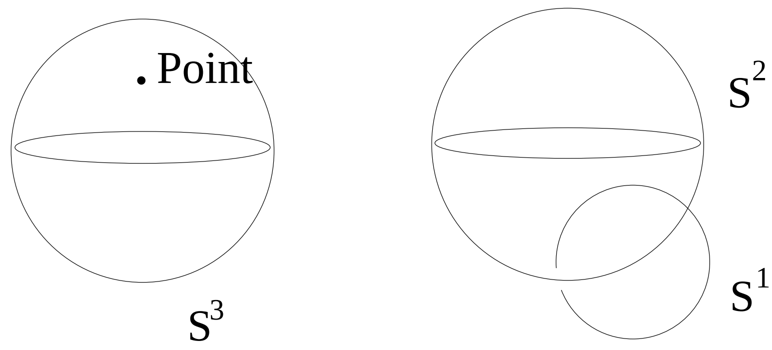
The possible configurations for Donaldson numbers in the case $k = 1$ depend on the relation

$$\sum_{j=1}^{l} d_i = 4l - 8k + 3 = 4l - 5$$

where d_i is the dimension of the submanifold Σ_i. When $l = 2$, we know that $d_1 + d_2 = 3$, so the only possible configurations are

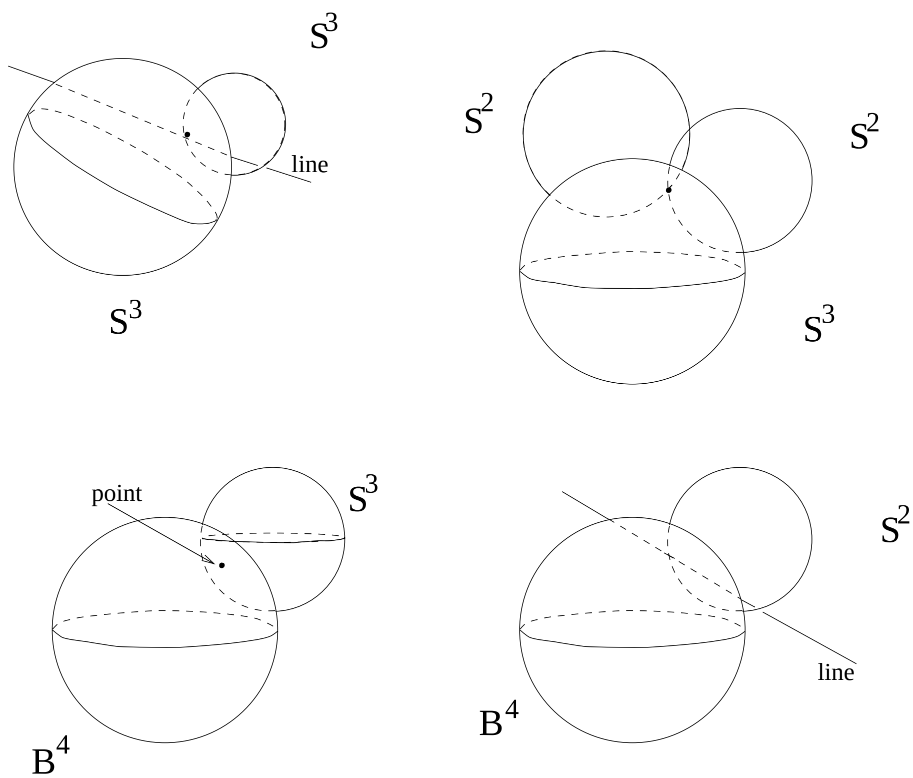
d_1	d_2
0	3
1	2

.



Configurations for $k=1$ $l=2$

For $l = 3$ we have the following linking configurations.



The configurations for $k=1$,$l=3$

Our next task will be to examine $k \geq 2$.

5.6 Is there any linking for $k \geq 2$?

It would be prudent to examine $\mathcal{P}(\Sigma_1, \ldots, \Sigma_l)$. Recall that

$$\begin{aligned}
&\mathcal{P}(\Sigma_1, \ldots, \Sigma_l)(\phi) \\
&= \int_{\mathbb{H}^{k-1}\times\mathcal{M}_1} \iota^*\phi \left(\mu^1(\Sigma_1) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_1)\right) \wedge \ldots \wedge \left(\mu^1(\Sigma_l) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_l)\right)
\end{aligned}$$

and let us look at the various configurations. Choose pairwise disjoint, submanifolds $\Sigma_1, \ldots \Sigma_l$ of $\mathbb{R}^4$ of dimensions $d_1, \ldots, d_l$ respectively. Then

$$\sum_{i=1}^{l} (4 - \mathrm{d}_i) = 8k - 3 \tag{12}$$

yields

$$\sum_{i=1}^{l} d_i = 4l - 8k + 3. \tag{13}$$

Definition 5.11 *Let K be a finite set. Then*

1. *we call an n-tuple I of elements of K an ordered subset of K whenever $I = (i_1, \ldots, i_p)$ we have $i_j \neq i_r$ for all $j \neq r$;*
2. *we say $n \in I = (i_1, \ldots, i_p)$ if there is j such that $i_j = n$;*
3. *we shall write $\#I = p$ if $I = (i_1, \ldots i_p)$;*
4. *we shall say that a collection $I_1, \ldots I_m$ of ordered subsets of K is a partition of K if for each $n \in K$ there is precisely one r such that $n \in I_r$.*

Now as we said earlier, each term in $\mathcal{P}$ is $\pm\mathrm{Don}_1(\Sigma_{I_1}) \prod_{j>1}^{p} \sharp(\Sigma_{I_i})$ for ordered subsets $I_1, \ldots, I_p$ that partition $\{1, \ldots, l\}$. For this to give a nonzero contribution to $\mathcal{P}$ we need I_1 to have at least 2 elements and each of the other I_i at least 1 element. Now I_1 must satisfy

$$\sum_{j\in I_1} (4 - \mathrm{d}_j) = 8(1) - 3 = 5 \tag{14}$$

that is

$$\sum_{j\in I_1} d_j = 4\#I_1 - 5. \tag{15}$$

Now we know that for each of $I_j, j > 1$ we need $\sharp(\Sigma_{I_j}) \neq 0$ so we must have for each $j > 2$,

$$\sum_{q \in I_j} (4 - d_q) = 4$$

that is

$$\sum_{q \in I_j} (d_q) = 4\#I_j - 4.$$

Now

$$\begin{aligned}
\sum_{j=1}^{l} d_j &= \sum_{n=1}^{p} \sum_{j \in I_n} d_j \\
&= \sum_{j \in I_1} d_j + \sum_{n=2}^{p} \sum_{j \in I_n} d_j \\
&= 4\#I_1 - 5 + \sum_{n=2}^{p} (4\#I_n - 4) \\
&= 4 \sum_{n=1}^{p} \#I_n - 5 - 4(p-1) \\
&= 4l - 4p - 1.
\end{aligned}$$

Thus we must have

$$4l - 4p - 1 = 4l - 8k + 3$$

i.e.

$$p = 2k - 1.$$

So k controls the number of ordered subsets of $\{1, \ldots, l\}$ that form a partition, moreover this number has to be odd.

However, if we examine the form of $\mathcal{P}$ more closely,

$$\begin{aligned}
&\mathcal{P}(\Sigma_1, \ldots, \Sigma_l)(\phi) \\
= &\int_{\mathbb{H}^{k-1} \times \mathcal{M}_1} \iota^* \phi \left(\mu^1(\Sigma_1) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_1) \right) \wedge \ldots \wedge \left(\mu^1(\Sigma_l) + \sum_{j=1}^{k-1} p_j^* PD(\Sigma_j) \right),
\end{aligned}$$

we see that each term is the product of sums of k terms, so for any ordered subsets $I_1, \ldots, I_p$ that partition $\{1, \ldots, l\}$ and give non-zero contribution to $\mathcal{P}$ must satisfy

$$p \leq k.$$

Thus we have

$$k \geq p = 2k - 1$$

which is impossible for $k > 1$. Hence $\mathcal{P}$ is a trivial topological distribution, assigning 0 to any set of pairwise disjoint, compact submanifolds of $\mathbb{R}^4$ and any test function ϕ. We have therefore proved

Theorem 5.12 *For $k \geq 2$, and any compactly supported ϕ there are no anomalies, i.e for $k \geq 2$*

$$\mathrm{Don}_k(\Sigma_1, \ldots, \Sigma_l)(\phi) = 0$$

for all pairwise disjoint, compact submanifolds $\Sigma_1, \ldots, \Sigma_l$.

We proved this for the resolution of the moduli space. Since the integral is identically 0 on the resolution, it must also be zero on the moduli space itself, thus agreeing with the infinite dimensional construction.

5.7 Concluding Remarks

Although we have had something of a disappointment that there is no linking number for $k > 1$ on the moduli space of instantons, nor on any resolutions, we have developed some potentially powerful techniques in finding formulæ for the cohomology of a hyperKähler reduction. One can hope that the technique for hyperKähler manifolds with boundaries may produce information about the topology of the higher instanton spaces by looking at the topology of the end and concluding that the Moduli space is a cone on this manifold. Also there may be something to be said about the perturbed moduli spaces with their relationship with the Seiberg-Witten equations.

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